

Equations

Navier-Stokes Equations discretised in space

We start with the incompressible Navier-Stokes equations in conservative form:

$$u_{,x} + v_{,y} = 0 \quad (1)$$

$$u_{,t} + (uu)_{,x} + (vu)_{,y} = -\frac{1}{\rho} p_{,x} + \nu(u_{,xx} + u_{,yy}) \quad (2)$$

$$v_{,t} + (uv)_{,x} + (vv)_{,y} = -\frac{1}{\rho} p_{,y} + \nu(v_{,xx} + v_{,yy}) - g \quad (3)$$

We want to discretise these equations in space with finite differences. We will apply central differences. The discretised parts of the Navier-Stokes equations are (without pressure term, which will be added in the pressure correction equation):

$$\begin{aligned} u_{,x} &= \frac{u_{i+1,j} - u_{i-1,j}}{2h_x}, & u_{,y} &= \frac{u_{i,j+1} - u_{i,j-1}}{2h_y} \\ v_{,y} &= \frac{v_{i,j+1} - v_{i,j-1}}{2h_y}, & v_{,x} &= \frac{v_{i+1,j} - v_{i-1,j}}{2h_x} \\ u_{,xx} &= \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h_x^2}, & u_{,yy} &= \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h_y^2} \\ v_{,yy} &= \frac{v_{i,j+1} - 2v_{i,j} + v_{i,j-1}}{h_y^2}, & v_{,xx} &= \frac{v_{i+1,j} - 2v_{i,j} + v_{i-1,j}}{h_x^2} \end{aligned} \quad (4)$$

We fill in equations (4) into equation (1):

$$\frac{u_{i+1,j} - u_{i-1,j}}{2h_x} + \frac{v_{ij+1} - v_{ij-1}}{2h_y} = 0 \quad (5)$$

Next we fill in equations (4) into equations (2) and (3):

$$u_{ij,t} + u_{ij} \frac{u_{i+1,j} - u_{i-1,j}}{2h_x} + v_{ij} \frac{u_{ij+1} - u_{ij-1}}{2h_y} = \nu \left(\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h_x^2} + \frac{u_{ij+1} - 2u_{i,j} + u_{ij-1}}{h_y^2} \right) \quad (6)$$

$$v_{ij,t} + u_{ij} \frac{v_{i+1,j} - v_{i-1,j}}{2h_x} + v_{ij} \frac{v_{ij+1} - v_{ij-1}}{2h_y} = \nu \left(\frac{v_{i+1,j} - 2v_{i,j} + v_{i-1,j}}{h_x^2} + \frac{v_{ij+1} - 2v_{i,j} + v_{ij-1}}{h_y^2} \right) - g \quad (7)$$

Navier-Stokes equations discretised in space and time

We start from (6) and (7), but first write it a little bit different:

$$u_{ij,t} = f_u(u, v) \quad (8)$$

$$v_{ij,t} = f_v(u, v)$$

We will discretise these equations in time with the use of the Adam Bashfort 2 method for time discretisation. We will first calculate a guess for the velocity and use the pressure from the old time level. After that we will update the pressure with the use of equation (1) and calculate the velocities for the new time level. The Adam Bashfort 2 scheme is given as:

$$\begin{aligned}
 u^* &= u^n + \frac{3}{2} f_u(u^n, v^n) - \frac{1}{2} f_u(u^{n-1}, v^{n-1}) \\
 v^* &= v^n + \frac{3}{2} f_v(u^n, v^n) - \frac{1}{2} f_v(u^{n-1}, v^{n-1})
 \end{aligned}
 \tag{9}$$

Or with Euler forward

$$\begin{aligned}
 u^* &= u^n + f_u(u^n, v^n) \\
 v^* &= v^n + f_v(u^n, v^n)
 \end{aligned}$$

Staggered pressure correction can be found in other scanned file.